Calogero-Sutherland gas of ultracold Bose atoms

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We show that the Calogero-Sutherland (C-S) gas, a famous exact soluble one-dimensional system with an inverse square long range interaction, can be realized by dimension reduction in a cold Bose atom system with a dipole-dipole interaction. Depending on the orientation of the dipoles, the effective interaction is either attractive or repulsive. The low-lying effective theory may be a Luttinger liquid when the exclusion statistics parameter λ may be well-defined. We hope that the C-S gas can be realized experimentally and the Luttinger liquid character can be observed.

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Introduction One-dimensional ultracold atom systems have shown amazing physical behaviors in a bunch of experiments in recent years. A phase transition from a one-dimensional Bose Einstein condensate(BEC) to a Mott insulator has been observed in a one-dimensional optical lattice [1]. The strong coupling limit of the one-dimensional gas with a contact interaction, known as the Tonks-Girardeau gas [2], has also reached in the ultracold Bose atoms [3].

Another kind of important one-dimensional systems with inverse square interaction was known as the Calogero-Sutherland (C-S) models [4, 5]. Nowadays the C-S models play a fundamental role in the contemporary theoretical physics. The eigen wave functions of these exactly soluble models may be explicitly expressed by symmetric polynomials [6, 7]. The quasi-particle excitations in these models obey exclusion statistics [8, 9, 10]. The low-lying effective theory is a Luttinger liquid [11, 12] characterized by the exclusion statistics parameter λ [7, 13]. It was shown [13, 14] that this Luttinger liquid is equivalent to a $c = 1$ conformal field theory on a circle with a radius $R = \sqrt{2/\lambda}$ [15]. The C-S model may be a universal Hamiltonian of some weakly disordered metals [16]. The models may also be used to describe the edge excitations in the fractional quantum Hall effect [17].

Besides mentioned above, there were already numerous works in studying the C-S type models [18], but until now a one-dimensional physical system with a real inverse square interaction still lacks experimental realization. Recently, a major progress in the cold atoms was that chromium atoms ^{52}Cr with a dipole-dipole interaction were turned into a BEC [19]. This provides an opportunity to induce an inverse square interaction by the dimension reduction. There already are many theoretical considerations for the BEC with dipole-dipole interactions in ultracold gases [20, 21]. To our knowledge, such a dimension reduction to the dipole interaction was not studied, which will be a main topic of this Letter.

Reducing to a single-band model With a very strong z-direction confinement, a cold atom BEC cloud has a pancake-like shape trapped in the x-y plane. Using an periodic optical potential along the x-direction, the pancake is incised into an array of cigar-like BECs. Apply an external field in (or perpendicular to) this x-y plane so that the dipoles in (perpendicular to) this plane are parallel. (See Fig. 1(b) or (c). We first neglect the slow varying harmonic trap in the x-direction.) This is a problem of the Bose atoms loaded into an optical lattice with high filling factor. For the alkali metal atoms, the system is described by multi-band Bose Hubbard model but can be mapped into an effective single-band model [22]. If the product of the average number N_0 in a BEC and the on-site repulsion U is much larger than the trapping potential along the y-direction while it is much smaller than the strength of the optical lattice, the wave function may be approximated by the product of the single atom ground state wave function in the x-direction and the BEC's in the y direction. In this approximation, the atomic field operator may be written as [22]

$$\psi(\mathbf{r}) \approx \sum_i a_i w(x - x_i)\Psi_{TF}(y) \tag{1}$$

where $w(x)$ is the Wannier function along the x-direction and $\Psi_{TF}(y)$ is the Thomas-Fermi wave function along the y-direction, which describes the ground state of atoms in each single site. $a_i^\dagger$ (a_i) is the creation (annihilation) operator for an atom located in the ground state at each site. The renormalized hopping t_R and interaction U_R can be estimated in the Thomas-Fermi approximation[22]. The hopping t may not be renormalized [23] while U_R is reduced from its bare value considerably due to the repulsive on-site interactions that spread out the condensate wave function. The consistent condition for this reduction of U_R is $l_p/a \ll N_0 \ll (\hbar\omega_y/\hbar\omega_p)^2\sqrt{2\pi}L_y/a$, where a is the s-wave scattering length, $\hbar\omega_y$ is the trapping energy in the y-direction, $L_y = \sqrt{\hbar/m\omega_y}$ is the effective length in the y-direction. l_p is the lattice spacing and $\hbar\omega_p = \hbar^2/ml_p^2$. These conditions may easily reach in experiment.

For the atom cloud with dipole-dipole interaction, besides the on-site interaction, there is another interaction between two parallel dipoles located in $\mathbf{r}$ and $\mathbf{r}'$, which

reads

$$V_d(\mathbf{r},\mathbf{r}') = d^2(1-3\cos^2\Theta)/R^3, \quad (2)$$

where Θ is the angle between $\mathbf{R}=\mathbf{r}-\mathbf{r}'$ and d is the magnitude of the dipole moment $\mathbf{d}$. For dipoles laying in the plane, Θ is shown in Fig. 1(a) while $\Theta=\pi/2$ for dipoles perpendicular to the plane. The Thomas-Fermi density of dipolar BEC has the same parabolic form as that in the short range interaction BEC. The dipole-dipole interaction only affects the central density and the Thomas-Fermi radius [20]. Therefore, the wave function factorization approximation is also valid if $\hbar\omega_y \ll N_0U[1+O(\epsilon_{dd})] \ll \hbar\omega_p$ where $U[1+O(\epsilon_{dd})]$ is the interacting energy scale [20]. ($\epsilon_{dd}<1$ for the practical system will be defined later.)

Consider a system with atom number $LN_0+N(N<L)$ where L the x-direction lattice size. The effective single-band lattice model can be obtained by keeping only the nearest neighbor hopping and expanding on-site energy to the second order near mean occupation N_0 [22, 23], i.e.,

$$\begin{aligned} H/N_0 &= -\sum_{\langle ij\rangle} t a_i^\dagger a_j + \frac{U_R}{2}\sum_i n_i(n_i-1) \\ &+ \sum_{i<j} U_{d,ij} n_i n_j, \end{aligned} \quad (3)$$

where $n_i = a_i^\dagger a_i - N_0$; $t=\int dx\ w^*(x-x_i)(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_p(x))w(x-x_j)$ for a pair of nearest neighbor sites. $U_R = \frac{4\pi\hbar^2 a}{mN_0}\int dx|w(x)|^4\int dy|\Psi_{TF}(y)|^4$ and $U_{d,ij} = \frac{1}{N_0}\int d\mathbf{r}d\mathbf{r}'|w(x-x_i)|^2|w(x'-x_j)|^2V_d(\mathbf{r},\mathbf{r}')|\Psi_{TF}(y)|^2|\Psi_{TF}(y')|^2$.

This is a homogeneous model by neglecting the trapping in the x-direction and describes the dynamics of the fluctuation around the mean occupation N_0. Adding back the harmonic trap with the frequency ω_x, the system has an effective size $L=\sqrt{\hbar/m\omega_x}$ in the x-direction . If the width of a single BEC is wider, the number N_0 of atom per site may vary site by site and N_0 in the center of trapping may be several times of N_0 in far away from the center. Within a cigar-like single dipolar BEC, the dipole-dipole repulsion is strong enough against rasing the width of the single BEC at the trapping center. Therefore, the factorization of the ground state wave is still valid and N_0 keeps an constant. The resulting inhomogeneous model is H in (3) added by a term $\sum_i V_{h,i}n_i$ where

$$V_{h,i} = \frac{1}{2}m\omega_x^2\int dx|w(x-x_i)|^2x^2 \quad (4)$$

is the harmonic trapping potential along the x-direction.

We now assume the trap in the y-direction is very weak such that $|\Psi_{TF}(y)|^2$ can be approximated by the average density ρ_0, which is consistent with another approximation we will use, i.e., assuming the condensate along the y-direction is infinitely long. Under these approximations, it is easy to integrate over y and y' and arrives at

$$\begin{aligned} U_{d,ij} &\approx G_{xy(z)}\int dxdx'|w(x-x_i)|^2|w(x'-x_j)|^2 \\ &\times \frac{1}{(x-x')^2}, \approx \frac{G_{xy(z)}}{(x_i-x_j)^2}. \end{aligned} \quad (5)$$

where $G_{xy}=-2d^2\bar{\rho}_0\cos^2\theta_0$ if the dipoles orient in the plane and $G_z=2d^2\bar{\rho}_0$ if dipoles is perpendicular to the plane. Here, θ_0 is the angle of the dipole with respect to the x axis (Fig.1(a)). If the on-site interaction is switched off, to prevent the BEC collapse due to the attractive dipole-diploe interaction in a single BEC, one takes $\cos^2\theta_0>\frac{1}{3}$ such that the dipoles in the y-direction are repulsive. Thus, we have a one-dimensional lattice model with the on-site and inverse square interactions between the single modes from the ground state. If the dipoles lay in the x-y plane (Fig. 1(b)), the effective interaction between those atoms in different sites is attractive while it is repulsive if the dipoles are perpendicular to the plane (Fig. 1(c)).

The stability condition of a single BEC at a given site is given by $\epsilon_{dd}=md^2/(3\hbar^2a)<1$ [19]. For ^{52}Cr, the experimental measured $\epsilon_{dd}=0.159\pm0.034$ [24]. However, the renormalized on-site interaction is reduced to [22]

$$U_R \sim \frac{4\pi\hbar^2 a}{m}\frac{l_P}{R_{TF}} \quad (6)$$

where R_{TF} is the Thomas-Fermi radius of the single BEC. On the other hand, d is not renormalizable as shown in (5). Thus, although $\epsilon_{dd}<1$, the renormalized on-site interaction may be much weaker than the dipole-diploe interaction since $\epsilon_{dd}R_{TF}/l_P \gg 1$ as $l_p/R_{TF}\ll 1$ for the cigar-like BEC we are considered. Therefore, in the effective one-dimensional single-band model, the dipole-dipole interaction dominates and the on-site interaction may be turned off.

Dilute gas limit and CS model In the dilute gas limit with $N/L\ll 1$, the N-particle system may be described by a continuous model because the dispersion $-t\cos kl_p/\hbar \sim k^2/(2m_R)$ with $m_R\sim\hbar^2/(tl_p^2)$. Neglecting the on-site interaction as argued above, the effective Hamiltonian reads

$$H_{CS} = \sum_{i=1}^N\left(-\frac{\hbar^2}{2m_R}\frac{d^2}{dx_i^2}+\frac{1}{2}m_R\omega_R^2x_i^2\right)+\sum_{i<j}\frac{G_{xy(z)}}{|x_i-x_j|^2}, \quad (7)$$

where the renormalized trapping potential is defined by $m\omega_x^2=m_R\omega_R^2$. The model described by the Hamiltonian (7) is the famous C-S model [4, 5] . A 'particle' at x_i in this continuous model corresponds to one more atom than N_0 at the lattice site i in the lattice model. The ground state wave function is given by

$$\Psi_{0,\lambda}(x_1,...,x_N) = \prod_{1\le j<k\le N}(x_j-x_k)^\lambda e^{-\frac{1}{2\hbar}m_R\omega_R\sum_j^N x_j^2}, \quad (8)$$

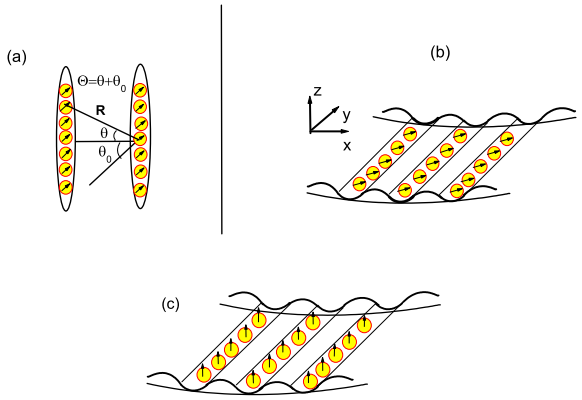

FIG. 1: (Color on-line) The trapped cold atom cloud in an optical lattice. (a) The parameters used in deriving eq. (5) for the case (b). (b) The dipoles are in the x-y plane. (c) The dipoles are perpendicular to the x-y plane. The angle θ is the same as shown in (a).

where λ is the solution of the equation $\lambda(\lambda-1) = -g_{xy} = (m_R/\hbar^2)G_{xy}$ or $= g_z = (m_R/\hbar^2)G_z$, respectively. The ground state energy $E_g = \frac{1}{2}N\hbar\omega_R(\lambda(N-1)+1)$.

For $\lambda = \lambda_{xy} = (1-\sqrt{1-4g_{xy}})/2$, if $g_{xy} < 1/4$ the ground state (8) is well-defined because $\lambda_{xy} > 0$ is real . Although the interaction is attractive, the particles in one dimension mutually exclude because the Jastraw factor in the ground state wave function forbids particles occupy the same position. The ground state energy E_g will be imaginary if $g_{xy} > \frac{1}{4}$. This means that the ground state is not stable when $g_{xy} > \frac{1}{4}$. The collapse of the ground state reflects the factorization of the wave function is false if the attraction between atoms is too strong. The system is back to a two-dimensional BEC if $\epsilon_{dd} < 1$ because the on-site repulsive is not renormalized.

For the repulsive coupling constant g_z, $\lambda_\pm = \frac{1\pm\sqrt{1+4g_z}}{2}$. For $-\frac{1}{2} < \lambda_- < 0$, i.e, $g_z < \frac{3}{4}$, the wave function is still square integrable. This implies if the repulsive between particles is not strong enough, the particles would like to gather together in the real space. Hence, the long wave length approximation used here is not valid and one has to be studied in the lattice model. We do not go this complicated case here. For a strong repulsion with $g_z > \frac{3}{4}$, (8) is no longer square integrable for $\lambda = \lambda_- < -\frac{1}{2}$. Instead, one should take $\lambda = \lambda_+ > 1.5$ in (8). This is the standard solution of the C-S model.

From now on we will focus on the C-S gas, i.e., $g_{xy} < \frac{1}{4}$ and $g_z > \frac{3}{4}$. For the C-S gas, all the excitation states may be constructed and may be expressed by so called hidden-Jack polynomials [25]. We can, in principle, calculate any physical observable. However, to directly measure these observable in a trapped gas is difficult. In experiments, a well-established technique is switching off all trapping potentials and letting the atom cloud freely expand. After a long enough time, record the image of the cloud. In this time of flight, the image of free atom cloud reflects the momentum distribution of the trapped atoms. Due to the harmonic trap, the momentum is no longer a good quantum number. Thus, the calculation from these exact eigen states in the thermodynamic limit becomes unwieldy. A frequently used approximation is the local density approximation. The starting point of this approximation is solving the homogeneous system in the thermodynamic limit, i.e., $\omega_R = 0$ and N and $L \to \infty$ while $\bar{\rho} = N/L$ is fixed. Solving the model in this limit yields to solve the thermodynamic limit of the Sutherland model with a period potential [5]

$$V = \frac{\hbar^2}{2m_R}\sum_{i\neq j}\frac{\pi^2\lambda(\lambda-1)}{L^2\sin^2[\pi(x_i-x_j)/L]}. \qquad (9)$$

The quantum states of the Sutherland model are labelled by the pseudo-momenta k_i which are determined by the Bethe ansatz equations $Lk_i = 2\pi I_i + \pi(1-\lambda)\sum_{j<i}\mathrm{sgn}(k_j-k_i)$. Due to the exclusion between the particles, the ground state is a pseudo-Fermi sea which is given by $\{k_j^0\} = \{-\frac{\pi\lambda}{L}(N-1), -\frac{\pi\lambda}{L}(N-3), ..., \frac{\pi\lambda}{L}(N-3), \frac{\pi\lambda}{L}(N-1)\}$. The pseudo-Fermi momentum is $k_F = \frac{\pi\lambda}{L}(N-1)$.

Notice that the situation in $\lambda < 1$ is very different from $\lambda > 1$. For $\lambda > 1$, in a physical momentum interval $\frac{2\pi}{L}$, there is only either one or zero *one* since the pseudo-momentum interval is $\frac{2\pi\lambda}{L} > \frac{2\pi}{L}$. But for $\lambda < 1$, as we have seen, the number of *ones* is larger than one in an interval $\frac{2\pi}{L}$. Such a cluster behavior of the particles stems from the attraction between the particles if $\lambda < 1$. To see this matter clearly, we calculate the single particle correlation function. In fact, all correlation functions and their asymptotic forms were calculated. (See, e.g., [7, 13].) In the thermodynamic limit, we need only know the asymptotic ones. For the particle $\Psi_\lambda^\dagger(x)$ with an exclusion statistics parameter λ, the single-particle correlation function in the $x \to \infty$ limit is given by $G(x,0;\lambda) \equiv \langle\Psi_\lambda(x,0)\Psi_\lambda^\dagger(0,0)\rangle \sim x^{-\lambda}$. For $\lambda = \lambda_+ > 1$, the momentum distribution of the single-particle is well-defined and continuous near the Fermi momentum $p_F = \hbar k_F/\lambda$, $G_{\lambda_+}(p) = G_{\lambda_+}(p_F) - \text{const.}\cdot\mathrm{sgn}(p-p_F)|p-p_F|^{\lambda_+-1}$, where p is the physical momentum with interval $\frac{2\pi}{L}$. Thus, it is a standard Luttinger liquid [11, 12].

However, for $\lambda = \lambda_{xy} < 1$, the momentum distribution of the single-particle is divergent at the Fermi point p_F, i.e., $G_{\lambda_{xy}}(p) \sim \mathrm{sgn}(p-p_F)|p-p_F|^{\lambda_{xy}-1}$. This divergence implies that there is no single particle propagation. The instability of this single particle Fermi surface comes from the attraction between the particles. The competition between this attraction and the zero-point motion may lead to a cluster propagating behavior as we have analyzed in the pseudo-Fermi sea. In the coordinate space, this cluster behaviors can be

seen as follows. Defining a composite particle operator [11] $\Psi_{n\lambda_{xy}}(x) = \lim_{a\to 0}[a^{-\frac{1}{2}n(n-1)}\Psi_{\lambda_{xy}}(x)\Psi_{\lambda_{xy}}(x+a)...\Psi_{\lambda_{xy}}(x+(n-1)a)]$, for an integer n, the momentum distribution of this composite particle near p_F is given by $G_{n\lambda_{xy}}(p) = G_{n\lambda_{xy}}(p_F) - \text{const.}\cdot\text{sgn}(p-p_F)|p-p_F|^{n\lambda_{xy}-1}$. The minimal n with $n\lambda_{xy} - 1 \geq 0$ is $n = [1/\lambda_{xy}] + 1$. However, the composite particle with $n' = [1/\lambda_{xy}] - 1$ will accompany with this $n = [1/\lambda_{xy}] + 1$ particle to insure the pseudo-Fermi sea is filled up but the former has a divergent distribution at the Fermi surface. A consistent choice is $n = l$ for $\lambda_{xy} = s/l$ and thus the exponent is $l\lambda_{xy} - 1 = s - 1$, e.g., $\lambda_{xy} = 2/5$, $s - 1 = 1$. This reflects the exclusion statistics of the quasi-particles.

Time of flight and local density approximation The time of flight experiment directly measure the momentum distribution in the trapped gas. In the y-direction along the cigar-like BEC, the dynamics is controlled by Bogoliubov quasiparticles excitation while in the x-direction, the dynamics is dominated by the Luttinger liquid described above. We hope this Luttinger liquid behavior may be observed by measuring the slope of the density profile at the Fermi surface in the x-direction. The density-density correlation function $\langle\hat{\rho}(x,0)\hat{\rho}(0,0)\rangle \sim x^{-2}$ corresponds to the shot noise in the time of flight image [26]. To experimentally observe this Luttinger liquid requires a very high resolving rate because there is background LN_0 atoms' image.

For $\omega_R \neq 0$, one may use the local density approximation. Defining the local Fermi momentum $p_F(x) = \sqrt{p_F^2 - m_R^2\omega_R^2 x^2/\lambda^2}$, the density distribution in the Thomas-Fermi approximation, which is valid in the thermodynamic limit, is defined by [27]

$$\begin{aligned} n(p) &= \frac{1}{2\pi}\int_{p_F(L)}^{p_F} dP \frac{\lambda}{m_R\omega_R}\frac{G_\lambda(p,P)P}{\sqrt{p_F^2 - P^2}} \\ &\approx \frac{\lambda^2}{4\pi\hbar m_R^2\omega_R^2}\frac{\bar{p}_F}{\sqrt{p_F^2 - \bar{p}_F^2}}G_\lambda(p,\bar{p}_F) \end{aligned} \qquad (10)$$

where $p_F(L) = \sqrt{p_F^2 - m_R\hbar\omega_R/\lambda}$ and $\bar{p}_F$ is in between p_F and $p_F(L)$. Unlike the short range interaction Bose gas, the long range interaction gas in one-dimension is more like the degenerate Fermi gas: Despite the spatial anisotropy of the trap, the momentum distribution is uniform in the k_x-direction [27].

Conclusions We have shown the possibility to realize the C-S gas in a cold atom system with dipole-dipole interaction. The ground state behaviors of the system in various parameters were discussed. The experimental implications to explore the properties of these ground states and low-lying excitations are suggested.

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